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NOTE

The Separator Theorem for Rooted Directed Vertex Graphs

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vertex graphs due to C. L. Monma and V. K. Wei (1986, *J. Combin. Theory Ser. B* **41**, 141–181), and present a modified separator theorem for the same class of graphs.

1. INTRODUCTION

Let F be a family of non-empty sets. An undirected graph G is an intersection graph for F if there is a one-to-one correspondence between the vertices of G and the sets in F such that two vertices in G are adjacent if and only if the corresponding sets have non-empty intersection. If F is a family of paths in an undirected tree T, then G is called an undirected vertex (UV) or a path graph. If F is a family of directed paths in a directed tree T, then G is called a directed vertex (DV) or a directed path graph. Note that a directed tree may have more than one vertex of indegree zero. A rooted directed tree is a directed tree having exactly one vertex of indegree zero. If F is a family of paths in a rooted directed tree T, then G is called a rooted directed vertex (RDV) graph.

Monma and Wei [1] presented a unified framework for characterizing UV, DV and RDV graphs. In [1], they presented characterizations of these graphs in terms of clique separator, which are called separator theorems.

In this note, we present a counter example to the separator theorem for RDV graphs due to Monma and Wei [1]. We, then, present a modified separator theorem for RDV graphs.

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2. PRELIMINARIES

Throughout the discussion our graph is assumed to be connected. Let $G=(V, E)$ be a graph. A set $C \subseteq V(G)$ is called a clique if the induced subgraph of G on C, denoted $G[C]$, is a maximal complete subgraph of G. Let $C(G)$ be the set of all cliques of G, and for each $v \in V(G)$, $C_v(G)$ denote the set of all cliques containing v.

Though in the definition of UV graphs and RDV graphs the trees are arbitrary, there exist trees satisfying nice properties, which are given in the following theorem.

THEOREM 2.1 (Monma and Wei [1], Clique Tree Theorem).

(a) *A graph $G=(V, E)$ is RDV if and only if there exists a rooted tree T with vertex set $C(G)$, such that for each $v \in V(G)$, $T[C_v(G)]$ is a directed path in T.*

(b) *A graph $G=(V, E)$ is UV if and only if there exist a tree T with vertex set $C(G)$, such that for every $v \in V(G)$, $T[C_v(G)]$ is a path in T.*

A tree satisfying Theorem 2.1 is called a clique tree for the graph it characterizes.

The following lemma gives more insight to the clique tree of UV graph.

LEMMA 2.2 (Monma and Wei [1, Proposition 7′]). *Let C be a clique in the UV graph G. If C is not a separator, then C is a leaf node (i.e. a node with degree 1) in every UV clique tree T of G.*

Next, we present the separator theorem of RDV graph due to Monma and Wei [1]. To this end, we need to introduce some new concepts.

If $G-C$ is disconnected by a clique into components $H_i=(V_i, E_i)$, $1 \leqslant i \leqslant r$, $r \geqslant 2$, then C is called a separating clique and $G_i=G[\{V_i \cup C\}]$, $1 \leqslant i \leqslant r$, $r \geqslant 2$ is said to be a separated graph of G with respect to C. Let C be a separating clique. Cliques which intersect C but not equal to C are called relevant cliques with respect to C.

In the following definitions, only relevant cliques are considered.

Let C_1 and C_2 be two cliques of G. We say (1) C_1 and C_2 are unattached, denoted $C_1 | C_2$, if $C_1 \cap C_2 \cap C=\varphi$; otherwise, they are attached, (2) C_1 dominates C_2, denoted $C_1 \geqslant C_2$, if $C_1 \cap C \supseteq C_2 \cap C$, (3) C_1 properly dominates C_2, denoted $C_1 > C_2$, if $C_1 \cap C \supset C_2 \cap C$, and (4) C_1 and C_2 are antipodal, denoted $C_1 \Leftrightarrow C_2$, if they are attached and neither dominates the other.

Let G_1 and G_2 be two separated graphs of G with respect to C. We say (1) G_1 and G_2 are unattached, denoted $G_1 \mid G_2$, if $C_1 | C_2$ for every clique C_1 in G_1 and for every clique C_2 in G_2; Otherwise they are attached, (2)

G_1 dominates G_2, denoted $G_1 \geqslant G_2$, if they are attached and for every clique C_1 in G_1, $C_1 \geqslant C_2$ for all cliques C_2 in G_2 or $C_1 \mid C_2$ for all cliques C_2 of G_2, (3) G_1 properly dominates G_2, denoted $G_1 > G_2$, if $G_1 \geqslant G_2$ but not $G_2 \geqslant G_1$, and (4) G_1 and G_2 are antipodal, denoted $G_1 \Leftrightarrow G_2$, if they are attached and neither dominates the other.

The following lemma gives an ordering of a collection of pair-wise non-antipodal subgraphs.

LEMMA 2.3 (Monma and Wei [1]). *Any collection of pair-wise non-antipodal separated graphs of a (general) graph can be arranged in such a way that $G_i > G_j$ implies $i < j$.*

Next, we present the separator theorem for RDV graphs due to Monma and Wei [1].

THEOREM 2.4 (Monma and Wei [1, Separator Theorem]). *G is an RDV graph if and only if each G_i is RDV and the G_i's can be 2-colored such that no antipodal pairs have the same color, and that in one color every subgraph has an RDV clique tree rooted at C, and that in the other color no two subgraphs are unattached and every subgraph (with one possible exception) has an RDV clique tree rooted at a relevant clique. The exceptional subgraph, should it exist, is dominated by every other subgraph of the same color, and it has an RDV clique tree in which the vertex C has out-degree zero.*

3. RESULTS

In this section, we present a counter example to Theorem 2.4. We, then, present a modified separator Theorem for RDV graph.

LEMMA 3.1. *The tree T of Fig. 3.1 is the unique UV clique tree of G of Fig. 3.1.*

Proof. Consider the graph G of Fig. 3.1. Now, $C = \{2, 3, 4, 5, 6\}$,$C_1 = \{1, 2, 3\}$, $C_2 = \{2, 4, 8\}$, $C_3 = \{2, 4, 5, 9\}$ $C_4 = \{9, 10, 5\}$, and $C_5 = \{5, 6, 7\}$ are the only cliques of G. Since, $C(G)$ corresponds to the vertex set of T, and for each $v \in V(G)$, $T[C_v(G)]$ is a path in T, T is a UV clique tree. Next, we prove that T is the unique UV clique tree for G. If possible, there is a UV clique tree T_1 of G such that $T_1 \neq T$. Since, the cliques C_1, C_2, C_4, and C_5 are non-separating cliques of G, by Lemma 2.2, these cliques will correspond to leaf vertices of T_1. So, CC_3 is an edge in T_1. Since $C_3(G) = \{C_1, C\}$ and $C_6(G) = \{C_5, C\}$, C_1 and C_5 will be adjacent to C in T_1.

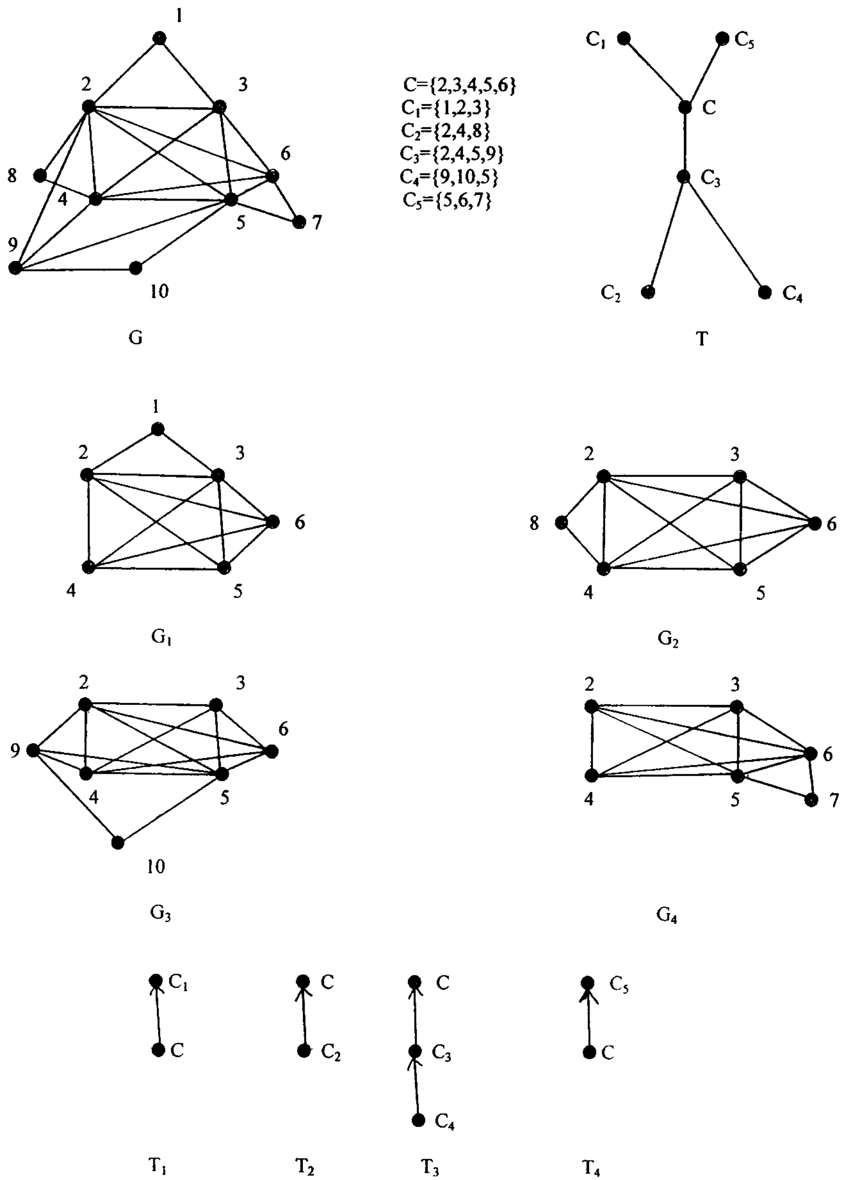

FIG. 3.1. A counterexample to Theorem 2.4.

Since, $C_9(G)=\{C_4, C_3\}$, C_4 will be adjacent to C_3. Again, since $C_2(G)=\{C_1, C, C_2, C_3\}$ and $T_1[C_2(G)]$ is a path in T_1, C_2 will be adjacent to C. So, $T_1=T$, which is a contradiction.

Hence, our lemma is proved. ∎

The following proposition gives a counter example to Theorem 2.4.

PROPOSITION 3.2. *The separated graphs G_1, G_2, G_3, and G_4 of G with respect to C satisfy the necessary conditions of Theorem* 2.4, *but G is not an RDV graph, where G, G_1, G_2, G_3, G_4, and C are as in Fig.* 3.1.

Proof. Color G_1 and G_4 by color 1, and G_2 and G_3 by color 2. It's easy to check that T_i is an RDV clique tree of G_i, $1 \leqslant i \leqslant 4$. It is now easy to verify that the separated graphs satisfy all the necessary conditions of Theorem 2.4.

Next, we claim that G is not an RDV graph. If possible, G is an RDV graph. Let T' be any RDV clique tree of G. Let T'' be the tree obtained from T' by ignoring the direction. Then, T'' is a UV clique tree of G. Since, by Lemma 3.1, T is the unique UV clique tree of G, $T''=T$. The cliques of G are $C=\{2, 3, 4, 5, 6\}$, $C_1=\{1, 2, 3\}$, $C_2=\{2, 4, 8\}$, $C_3=\{2, 4, 5, 9\}$, $C_4=\{9, 10, 5\}$, and $C_5=\{5, 6, 7\}$. Since, $C_2(G)=\{C_1, C_2, C_3, C\}$, C_1, C, C_3, C_2 will be a directed path in T'. This implies, the root of T' is either C_1 or C_2. Again, $C_5(G)=\{C_1 C_3, C_4, C_5\}$. So, C_5, C, C_3, C_4 will be directed path in T'. This is impossible, since the root of T' is either C_1, or C_2. So we have a contradiction. Hence G is not an RDV graph. ∎

So, the conditions, stated in Theorem 2.4, are not sufficient for a graph to be an RDV graph. Hence, the proof of Theorem 2.4 has a flaw. In the proof of the sufficiency of Theorem 2.4 (see [1], page 173, line 15) it is mentioned that relevant cliques form the path from the root to the clique C in the RDV clique tree for each subgraph of color 2. This statement is not true if a separated graph having color 2 has two unattached relevant cliques. Again, the relevant cliques of all the separated graphs having color 2 must form a path in the tree obtained from merging the RDV clique trees (see the method of merging in [1]) of all separated graphs of color 2. This is not possible if there exists two unattached relevant cliques belonging to the same separated graph having color 2 or two different separated graphs having color 2.

However, if we replace the condition "that in the other color no two subgraphs are unattached" of Theorem 2.4 by a stronger condition "that in the other color no two relevant cliques are unattached", then the modified conditions are sufficient for a graph to be an RDV graph. Note that, the new condition is a stronger condition than the old condition.

Finally, we present the modified separator theorem for RDV graph. The proof of Theorem 3.3 goes along the same lines of that of Theorem 2.4 (see [1]).

THEOREM 3.3 [Modified Separator Theorem]. *G is an RDV graph if and only if each G_i is RDV, and the G_i's can be two-colored such that no antipodal pairs have the same color, and that in one color every subgraph has an RDV clique tree rooted at C, and that in the other color no two relevant cliques are unattached, and every subgraph (with one possible exception) has an RDV clique tree rooted at a relevant clique. The exceptional subgraph, should it exist, is dominated by every other subgraph of the same color, and it has an RDV clique tree in which the vertex C has out degree zero.*

Proof. Necessity: Let T be any RDV clique tree for G. If C is the root of T, then color all the subgraphs with color one and it is easy to see that the G_i's satisfy our Theorem. So assume that C is not the root of T. Color a separated graph by color 1 if it lies in an out going branch with respect to C; Otherwise, color it by color 2. Note that antipodal graphs receive different colors in the above coloring.

Let T^* be the subtree of T rooted at C. Then T^* is an RDV clique tree for G^* where $G^* = \{G_i \mid G_i \text{ is colored } 1\}$. For every subgraph G_i having color 1, an RDV clique tree T_i rooted at C can be easily constructed from T^*. Next, we consider the graphs having color 2.

The vertices corresponding to the relevant cliques form a contiguous part of the path from the root to C. Hence, no two relevant cliques are unattached. Let C_i^* be the relevant clique of G_i that is closest to the root. Let T_i be the subtree of T rooted at C_i^*. From T_i it is easy to construct in the same way as in color 1, an RDV clique tree T_i^* for G_i rooted at C_i^*. The only possible exception is the subgraph containing the root clique, say G_i'. Note that exception occurs exactly when the root clique is not a relevant clique. In this case G_i' is dominated by every other separated graphs having color 2 and the tree T_i' obtained from T by removing T^* is an RDV clique tree for G_i'. A clique tree for G_i' with C as a leaf can be easily derived.

Sufficiency: Let the separated graphs be colored in two colors, say color 1 and color 2, satisfying the conditions of our theorem. The RDV clique trees rooted at C for the subgraphs with color 1 can be glued together by the following construction rule R_1 to form an RDV clique tree T^* rooted at C for the subgraph G^*, where $G^* = \cup G_i$, $G_i \in S_1$, where $S_1 = \{i \mid G_i \text{ is given color } 1\}$.

Rule R_1: We will suggest a recursive construction rule. Since no two separated graphs having color 1 are antipodal, by Lemma 2.3, G_i's, $i \in S_1$ can be ordered such that $G_i > G_j$ implies $i < j$. Let $G_1, G_2, \ldots, G_r$ be this ordering. Let T^i be the RDV clique tree for $G_1 \cup G_2 \cup \cdots \cup G_i$ rooted at C. If G_{i+1} is unattached to every G_j, $1 \leqslant j \leqslant i$, then merge the root C of the clique tree of G_{i+1} with the root C of T^i to form T^{i+1}. Otherwise, let k be the largest index such that $G_k \geqslant G_{i+1}$. Let C_k be the clique of G_k which is farthest from C and which dominates every relevant cliques of G_{i+1}. Now merge the root of the clique tree of G_{i+1} with C_k and call the new vertex C_k. The tree so obtained is T^{i+1}. It is easy to see that T^{i+1} is an RDV clique tree for $G_1 \cup G_2 \cup \cdots \cup G_{i+1}$.

In the RDV clique tree for each subgraph of color 2, relevant cliques form the path from the root to the clique C. Hence, these trees can be glued together by the same method to construct a clique tree T'' rooted at a relevant clique. The clique tree of the exceptional subgraph can be glued with T'' to form an RDV clique tree T''' in which C is a leaf. Then T' and T''' can be glued at the clique C to obtain an RDV clique tree T for G. ∎

REFERENCE

1. C. L. Monma and V. K. Wei, Intersection graphs of paths in a tree, *J. Combin. Theory Ser. B* **41** (1986), 141–181.